

On crystal operators in Lusztig's parametrizations and string cone defining inequalities

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ABSTRACT : Let \mathbf{w}_0 be a reduced expression for the longest element of the Weyl group, adapted to a quiver of type A_n . We compare Lusztig's and Kashiwara's (string) parametrizations of the canonical basis associated with \mathbf{w}_0 . Crystal operators act in a finite number of patterns in Lusztig's parametrization, which may be seen as vectors. We show this set gives the system of defining inequalities of the string cone constructed by Gleizer and Postnikov. We use combinatorics of Auslander-Reiten quivers, and as a by-product we get an alternative enumeration of a set of inequalities defining the string cone, based on hammocks.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra corresponding to a Dynkin diagram D of type A_n, D_n, E_n , defined over $\mathbb{C}(q)$, q being an indeterminate. There is a braid group action on $U_q(\mathfrak{g})$ which enables to construct PBW bases [15] of the positive part $U_q(\mathfrak{n}^+)$ of $U_q(\mathfrak{g})$. Such a basis $\mathcal{P}_{\mathbf{w}_0}$ depends on the choice of a reduced expression \mathbf{w}_0 of the longest element w_0 of the Weyl group W associated with D . However, it was observed by Lusztig that the $\mathbb{Z}[q^{-1}]$ -module \mathcal{L} generated by $\mathcal{P}_{\mathbf{w}_0}$ is independent of \mathbf{w}_0 . Furthermore, the image of $\mathcal{P}_{\mathbf{w}_0}$ under the projection $\pi : \mathcal{L} \longrightarrow \mathcal{L}/q^{-1}\mathcal{L}$ is a \mathbb{Z} -basis B of $\mathcal{L}/q^{-1}\mathcal{L}$, which is again independent of \mathbf{w}_0 . There is a unique basis of \mathcal{L} , which is invariant under the \mathbb{C} -algebra involution of $U_q(\mathfrak{n}^+)$ preserving the generators of $U_q(\mathfrak{n}^+)$, and sending q to q^{-1} , and whose image under π is B . This is the canonical basis \mathcal{B}_{can} of Lusztig and Kashiwara [15], [12]. This basis is in one-to-one correspondence with any PBW basis $\mathcal{P}_{\mathbf{w}_0}$, yet is independent of the choice of \mathbf{w}_0 . It has with many remarkable properties. The basis \mathcal{B}_{can} is however difficult to compute for arbitrary Dynkin diagrams.

Let I denote the set of vertices of D . Kashiwara introduced crystal operators $\tilde{e}_i, \tilde{f}_i, i \in I$ on $U_q(\mathfrak{n}^+)$. These allow to construct the crystal graph $B(\infty)$ which serves as a combinatorial skeleton of \mathcal{B}_{can} . Its vertices are the elements of B , and its edges are induced by the action of crystal operators on \mathcal{B}_{can} . The crystal limit $b \mapsto b \bmod q^{-1}\mathcal{L}$ establishes a one-to-one correspondence between \mathcal{B}_{can} and vertices of $B(\infty)$, which allows to extract important combinatorial information from \mathcal{B}_{can} to the level of $B(\infty)$. The crystal graph $B(\infty)$ may be defined by purely combinatorial means, and provides important data for the study of finite-dimensional representations of $U_q(\mathfrak{g})$.

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The crystal limit also establishes a one-to-one correspondence between members of a basis $\mathcal{P}_{\mathbf{w}_0}$ and the vertices of $B(\infty)$. A PBW monomial is defined by an N -tuple of positive integers, where N is the number of positive roots in the root system associated to D . One thus gets an indexation of B by \mathbb{N}^N . This is the Lusztig's parametrization with respect to \mathbf{w}_0 .

Its advantage lies in the simple indexing set for B . The action of a crystal operator \tilde{e}_i is easy to describe when the reduced expression \mathbf{w}_0 starts with the simple reflection s_i . It is however difficult to give, for a fixed \mathbf{w}_0 , the action of all the operators \tilde{e}_i , $i \in I$ in the same time, due to the complexity of the passage formulas [4] between Lusztig parametrizations. This was done by Reineke [18] for reduced expressions \mathbf{w}_0 adapted to quivers Q of ADE type verifying a particular homological condition (L) (detailed in section 2). The Hall algebra construction [20] of $U_q(\mathfrak{n}^+)$ allows Reineke to study the crystal operators \tilde{e}_i , $i \in I$ using the representation theory of finite dimensional algebras.

Kashiwara showed [13] that given \mathbf{w}_0 , there is an elementary construction of $B(\infty)$ depending on \mathbf{w}_0 , known as Kashiwara's embedding. The vertices B of $B(\infty)$ are indexed a set $\mathcal{S}_{\mathbf{w}_0}$ of specific N -tuples of integers, known as string parameters. The action of crystal operators on B is easy to describe, as it depends only on the Cartan matrix of D . However it is a complex problem to describe the parameter set $\mathcal{S}_{\mathbf{w}_0}$. It is the set of integer points of a polyhedral cone $\mathcal{C}_{\mathbf{w}_0}$ [14], [4]. A system of inequalities defining $\mathcal{C}_{\mathbf{w}_0}$ was given by Littelmann [14] for particular reduced expressions with a good structure. Such a set of inequalities, for arbitrary \mathbf{w}_0 , was given by Gleizer and Postnikov [10] in A_n case, and Berenstein and Zelevinsky [4] for all finite Dynkin types.

Any cone inequality may be seen as $\mathbf{a} \cdot \mathbf{x} \geq 0$ where $\mathbf{a} \in \mathbb{R}^N$ is a vector orthogonal to the hyperplane of the inequality. Thus a polyhedral cone may be seen as being defined by a finite set of vectors. The methods of [10] and [4] construct respectively sets of vectors $K_{\mathbf{w}_0}^{GP}$, $K_{\mathbf{w}_0}^{BZ}$ with integer coordinates, defining $\mathcal{C}_{\mathbf{w}_0}$. A move associated to a crystal operator \tilde{e}_i in a given parametrization, is a vector \mathbf{v} appearing as the difference between \mathbf{t} and $\tilde{e}_i \mathbf{t}$ for some N -tuple \mathbf{t} . We shall denote by $L_{\mathbf{w}_0}$ the set of all possible moves, for all \tilde{e}_i , $i \in I$, in the Lusztig parametrization with respect to \mathbf{w}_0 . Reineke's construction allows, for the reduced expressions \mathbf{w}_0 for which it valid, to describe the set $L_{\mathbf{w}_0}$ in terms of the Auslander-Reiten quiver Γ_Q of Q .

The main result of this article (Theorem 2.4) is that for \mathbf{w}_0 adapted to a quiver Q of type A_n , one has $K_{\mathbf{w}_0}^{GP} = L_{\mathbf{w}_0}$. Thus the problem of constructing the cone $\mathcal{C}_{\mathbf{w}_0}$ seems to be the the same as the one of describing action of operators \tilde{e}_i in Lusztig's parametrization for \mathbf{w}_0 . We conjecture the set of Lusztig moves $L_{\mathbf{w}_0}$ defines $\mathcal{C}_{\mathbf{w}_0}$ for reduced expression adapted to quivers Q of ADE type, under the assumption that condition (L) required by Reineke on Q is verified. We give in the last section, a D_n example.

A by-product of the main theorem is that Auslander-Reiten quivers allow to compute a set of defining inequalities of $\mathcal{C}_{\mathbf{w}_0}$. The combinatorics involved is that of hammocks, introduced by Brenner [6]. This provides an alternative to methods given in [10], [4].

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2. Combinatorial models of the canonical basis

Let Φ be the root system corresponding to the Dynkin diagram D and $(,)$ the Cartan scalar product over $\mathbb{R}\Phi$. We shall denote by $\alpha_i, i \in I$ the set of simple roots, $\omega_i, i \in I$ the set of fundamental weights, and by $s_i, i \in I$ the simple reflections inside W . We shall fix all through this section, a reduced expression $\mathbf{w}_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ of the longest element w_0 of W . This expression induces the reflection ordering $\preceq_{\mathbf{w}_0}$ on the set of positive roots Φ^+ , a total ordering given by $\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} s_{i_2} \dots s_{i_{N-1}}(\alpha_{i_N})$.

We shall use here the conventions of [4], concerning the quantized enveloping algebra $U_q(\mathfrak{g})$ associated to D . It is generated by the set $e_i, f_i, k_i^{\pm 1}, i \in I$ subject to relations derived from the Cartan matrix C . The reader may find details of the defining relations in [4], section 3.1. We recall in particular that $[n]_q!$ denotes the q -factorial of n , and that the n -th divided power of an element $x \in U_q(\mathfrak{g})$ is given by $x^{(n)} = x^n / [n]_q!$.

The positive part $U_q(\mathfrak{n}^+)$ is the subalgebra generated by the $e_i, i \in I$. It admits a grading by $Q^+ = \bigoplus \mathbb{N}\alpha_i$ obtained by putting $\deg(e_i) = \alpha_i$. Given an arbitrary $\gamma \in Q^+$, the weight space $U_q(\mathfrak{n}^+)_{\gamma}$ is the $\mathbb{C}(q)$ -vector space of elements of a degree γ . All weight spaces of $U_q(\mathfrak{n}^+)$ are finite dimensional.

The braid group acts on $U_q(\mathfrak{g})$ by automorphisms $T_i, i \in I$ (noted $T'_{i,-1}$ in part VI of [16]). We refer again to [4] section 3 for a detailed definition. For every $k = 1, 2, \dots, N$, $E_{\beta_k} := T_{i_1} T_{i_2} \dots T_{i_{k-1}}(E_{i_k})$ is an element of $U_q(\mathfrak{n}^+)$ of weight β_k . Any given N -tuple $\mathbf{t} = (t_1, \dots, t_N)$ of positive integers defines the **PBW monomial**

$$p_{\mathbf{w}_0}(\mathbf{t}) := E_{\beta_1}^{(t_1)} E_{\beta_2}^{(t_2)} \dots E_{\beta_N}^{(t_N)}.$$

The set of all such monomials, $\mathcal{P}_{\mathbf{w}_0} := \{p_{\mathbf{w}_0}(\mathbf{t}) \mid \mathbf{t} \in \mathbb{N}^N\}$, forms the **PBW-basis** of $U_q(\mathfrak{n}^+)$ associated to the reduced expression \mathbf{w}_0 .

Theorem 2.1 [15]

For every monomial $p_{\mathbf{w}_0}(\mathbf{t})$, there is one and only one $b \in \mathcal{B}_{can}$ such that $b = p_{\mathbf{w}_0}(\mathbf{t}) \bmod q^{-1}\mathcal{L}$.

The crystal operators $\tilde{e}_i, \tilde{f}_i, i \in I$ where introduced by Kashiwara [12] for the negative part $U_q(\mathfrak{n}^-)$ of $U_q(\mathfrak{g})$. As $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{n}^+)$ are isomorphic as algebras, this construction may be carried over to $U_q(\mathfrak{n}^+)$.

Given $i \in I$, there is a locally nilpotent action θ_i over $U_q(\mathfrak{n}^+)$ defined by :

$$\theta_i(1) = 0, \quad \forall x \in U_q(\mathfrak{n}^+) : \theta_i(e_j x) = q^{(\alpha_i, \alpha_j)} e_j \theta_i(x) + \delta_{i,j} x.$$

One has $U_q(\mathfrak{n}^+) = \bigoplus_{n \in \mathbb{N}} e_i^{(n)} \ker \theta_i$, and $\ker \theta_i$ is compatible with the weight graduation of $U_q(\mathfrak{n}^+)$. One chooses a weight vector basis Ξ_i of $\ker \theta_i$, and defines for each $\mathbf{v} \in \Xi_i$ and any $n \in \mathbb{N}$:

$$\begin{aligned} \tilde{e}_i(e_i^{(n)} v) &:= e_i^{(n+1)} v, \\ \tilde{f}_i(e_i^{(n)} v) &:= \begin{cases} e_i^{(n-1)} v & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases} \end{aligned}$$

This leads to well defined operators \tilde{e}_i, \tilde{f}_i over $U_q(\mathfrak{n}^+)$, which do not depend on the initial choice of Ξ_i .

The crystal operators $\tilde{e}_i, \tilde{f}_i, i \in I$ preserve \mathcal{L} , and hence induce an action over $\mathcal{L}/q^{-1}\mathcal{L}$. A key feature of the canonical basis is its good behaviour under this action. For any $b \in \mathcal{B}_{can}$, $\tilde{e}_i b = b' \bmod q^{-1}\mathcal{L}$, and $\tilde{f}_i b$ is either $0 \bmod q^{-1}\mathcal{L}$ or $\tilde{f}_i b = b' \bmod q^{-1}\mathcal{L}$, where b', b'' are other elements of \mathcal{B}_{can} . We see the image B of \mathcal{B}_{can} inside $\mathcal{L}/q^{-1}\mathcal{L}$ becomes endowed with a structure of a coloured graph, the arrows being valuated by the operators $\tilde{e}_i, \tilde{f}_i, i \in I$. This is the crystal graph $B(\infty)$ of Kashiwara [12].

Our discussion in the introduction and Theorem 2.1 above lead to a one-to-one correspondence $\varphi_{\mathbf{w}_0} : \mathbb{N}^N \longrightarrow B, \mathbf{t} \mapsto \mathbf{p}_{\mathbf{w}_0}(\mathbf{t}) \bmod q^{-1}\mathcal{L}$. This is **Lusztig's parametrization** with respect to \mathbf{w}_0 . Under this identification, we may consider the crystal operators \tilde{e}_i, \tilde{f}_i as acting on \mathbb{N}^N . We shall call a vector $\mathbf{l} \in \mathbb{Z}^N$ a **Lusztig move of type i** with respect to \mathbf{w}_0 , if there exists $\mathbf{t} \in \mathbb{N}^N$ such that $\tilde{e}_i \mathbf{t} = \mathbf{t} + \mathbf{l}$. Recall $L_{\mathbf{w}_0}$ denotes the set of all possible Lusztig moves, for all types $i \in I$.

Example :

Consider A_2 case, $\mathbf{w}_0 = s_1 s_2 s_1$. For a given $\mathbf{t} = (t_1, t_2, t_3)$ one has :

$$\begin{aligned} \tilde{e}_1(t_1, t_2, t_3) &= (t_1 + 1, t_2, t_3) \\ \tilde{e}_2(t_1, t_2, t_3) &= \begin{cases} (t_1 - 1, t_2 + 1, t_3) & \text{if } t_1 > t_3 \\ (t_1, t_2, t_3 + 1) & \text{if } t_1 \leq t_3. \end{cases} \end{aligned}$$

One sees there is only one Lusztig move of type 1, $\mathbf{l}_1 = (1, 0, 0)$ and two Lusztig moves of type 2, $\mathbf{l}_2 = (-1, 1, 0)$, $\mathbf{l}_3 = (0, 0, 1)$. We get $L_{\mathbf{w}_0} = \{(1, 0, 0), (-1, 1, 0), (0, 0, 1)\}$.

Let Q be a fixed quiver obtained by orienting the Dynkin diagram D . Following [5], we call a vertex i of Q a sink, if there are only arrows entering it. We denote in that case by $s_i Q$ the quiver obtained by reversing the arrows whose end is i , into arrows exiting i , thus transforming i into a source. A reduced expression $\mathbf{w}_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ is adapted to Q if and only if i_1 is a sink of Q , i_2 a sink of $s_{i_1} Q$, i_3 a sink of $s_{i_2} s_{i_1} Q$ and so on. Such an expression always exists for a given Q .

Let us denote by $\mathbb{C}Q$ the path algebra of Q over \mathbb{C} . The category $\text{mod } \mathbb{C}Q$ of finite dimensional left modules has simple objects S_i which are indexed by I . We shall say, following Reineke [18], that the quiver Q verifies **condition (L)**, if for every indecomposable module $X \in \text{mod } \mathbb{C}Q$, and every $i \in I$, one has $\dim \text{Hom}(X, S_i) \leq 1$. This condition is verified for any quiver of type A_n , and at least one quiver of each of the types D_n, E_6, E_7 ([18] Appendix). Under this condition, the action of crystal operators in a Lusztig parametrization with respect to \mathbf{w}_0 adapted to Q may be described in terms of the category $\text{mod } \mathbb{C}Q$, and the set $L_{\mathbf{w}_0}$ may be obtained [18]. We postpone the details to section 3.

We continue to fix the same $\mathbf{w}_0 = s_{i_1} \dots s_{i_N}$. We refer the reader to Kashiwara [13], and [11] Chapter 5 for details on crystal theory. Kashiwara's elementary construction of $B(\infty)$ uses the Cartan matrix $C = (c_{i,j})$ in order to define operators $\tilde{e}_i, \tilde{f}_i, i \in I$ acting over \mathbb{N}^N as below ([11] 5.2.5, 6.1.15) :

Fix $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{N}^N$. For $k = 1, 2, \dots, N$, define $r_k := a_k + \sum_{1 \leq j < k} c_{i_j, i_k} a_j$.

Given $i \in I$, consider $\xi_i = \max_{i_k=i} r_k$. Let k_1 be the first position where this maximum

is attained, k_2 the last. Then :

$$\begin{aligned}\tilde{e}_i(\mathbf{a}) &= (a_1, \dots, a_{k_2-1}, a_{k_2} + 1, a_{k_2+1}, \dots, a_N). \\ \tilde{f}_i(\mathbf{a}) &= \begin{cases} (a_1, \dots, a_{k_1-1}, a_{k_1} - 1, a_{k_1+1}, \dots, a_N) & \text{if } a_{k_1} \geq 1, \\ 0 & \text{if } a_{k_1} = 0. \end{cases}\end{aligned}$$

Theorem 2.2 : *Kashiwara's embedding* [12]

Let \mathcal{N} denote \mathbb{N}^N with the action of operators $\tilde{e}_i, \tilde{f}_i, i \in I$ given above.

a) There is an embedding $\psi_{\mathbf{w}_0} B \hookrightarrow \mathbb{N}^N$, sending the graph $B(\infty)$ isomorphically onto the subgraph of \mathcal{N} generated out of $\mathbf{u}_\infty := (0, 0, \dots, 0)$ by applying the operators $\tilde{e}_i, i \in I$.

b) The image of $B(\infty)$ consists of those $\mathbf{a} = (a_1, a_2, \dots, a_N) \in \mathbb{N}^N$ verifying

$$\forall k = 1 \dots N : \quad \tilde{f}_{i_k}(\tilde{e}_{i_{k-1}}^{a_{k-1}} \tilde{e}_{i_{k-2}}^{a_{k-2}} \dots \tilde{e}_{i_1}^{a_1} \mathbf{u}_\infty) = 0.$$

Elements $\mathbf{a} := (a_1, a_2, \dots, a_N)$ in b) above are called **string parameters** [3]. The parametrization the set B of vertices $B(\infty)$ obtained through Theorem 2.2 is **Kashiwara's parametrization** with respect to \mathbf{w}_0 . Recall we denote its indexing set $\text{Im } \psi_{\mathbf{w}_0}$ by $\mathcal{S}_{\mathbf{w}_0}$.

Remarks

i) Kashiwara works with the negative part $U_q(\mathfrak{n}^-)$ of $U_q(\mathfrak{g})$. The definition above is the transfer of his construction to $U_q(\mathfrak{n}^+)$, which amounts to exchanging the roles of \tilde{e}_i and \tilde{f}_i at the level of $B(\infty)$.

ii) The definition of Kashiwara's embedding imposes a reversal of order in the definition of a string, namely (a_1, a_2, \dots, a_N) in our convention, corresponds to $(a_N, a_{N-1}, \dots, a_1)$ in [3].

Theorem 2.3 [14], [10], [4]

The set $\mathcal{S}_{\mathbf{w}_0}$ is the set of integer points of a polyhedral cone $\mathcal{C}_{\mathbf{w}_0}$, that is, there exists a finite set of vectors $K_{\mathbf{w}_0} \subset \mathbb{Z}^N$ such that

$$\mathcal{S}_{\mathbf{w}_0} = \{\mathbf{a} \in \mathbb{N}^N \mid \forall \mathbf{k} \in K_{\mathbf{w}_0}, \mathbf{a} \cdot \mathbf{k} \geq 0\}.$$

Example

Consider type A_2 , and $\mathbf{w}_0 = s_1 s_2 s_1$. It is easy to compute the image of Kashiwara's embedding directly out of its definition above. One obtains the well known result

$$\mathcal{C}_{\mathbf{w}_0} = \{(a_1, a_2, a_3) \mid 0 \leq a_1 \leq a_2, 0 \leq a_3\}.$$

One may choose as a defining set for $\mathcal{C}_{\mathbf{w}_0}$, the set $K_{\mathbf{w}_0} = \{(1, 0, 0), (-1, 1, 0), (0, 0, 1)\}$, which is equal to both $K_{\mathbf{w}_0}^{GP}$ and $K_{\mathbf{w}_0}^{BZ}$.

Main Theorem 2.4

Let \mathbf{w}_0 be a reduced expression adapted to a quiver of type A_n , and $K_{\mathbf{w}_0}^{GP}$ the set given by Gleizer and Postnikov ([10] section 5). Then

$$K_{\mathbf{w}_0}^{GP} = L_{\mathbf{w}_0}.$$

Let us observe that in Lusztig's parametrization, the parameter set is the set of integer points of the cone $(\mathbb{R}^+)^N$, consisting of vectors with positive coordinates. This cone may be seen as being defined by the natural basis $E = \{\mathbf{e}_1, \mathbf{e}_2 \dots \mathbf{e}_N\}$ of \mathbb{R}^N . It is easy to see, by the definition of Kashiwara's embedding, that the set E is the set of moves of crystal operators \tilde{e}_i , $i \in I$ in Kashiwara's parametrization according to \mathbf{w}_0 . We have therefore a full symmetry between Lusztig's and Kashiwara's parametrizations, the set of vectors defining the parameters set in one picture being equal to the set of crystal operators moves in the other.

Conjecture

Let Q be a quiver of type ADE satisfying Reineke's condition (L). Let \mathbf{w}_0 be adapted to it. Then $L_{\mathbf{w}_0}$ is a defining set for $\mathcal{C}_{\mathbf{w}_0}$.

We state this conjecture on the basis of some computer testing, using the sets $K_{\mathbf{w}_0}^{BZ}$. We give in section 7 a detailed D_4 example. The conjecture might be valid in a larger scope, even beyond reduced expressions adapted to quivers. However one faces a breakdown of many nice properties, enjoyed by reduced expressions adapted to quivers verifying condition (L).

3. Auslander-Reiten quivers and Lusztig's moves

Fix Q a quiver of type ADE satisfying condition (L), \mathbf{w}_0 adapted to it and $\beta_1, \beta_2, \dots, \beta_N$ the reflection ordering it defines. The category $\text{mod } \mathbb{C}Q$ is equivalent to that of finite dimensional representations of Q . A module M in $\text{mod } \mathbb{C}Q$ may be seen as a family $(V_i)_{i \in I}$ of finite dimensional \mathbb{C} -vector spaces, together with linear mappings $f_{i,j} : V_i \longrightarrow V_j$ corresponding to the arrows $i \longrightarrow j$ of Q . The dimension vector of M is the element of Q^+ given by $\mathbf{d}_M := \sum_{i=1}^n (\dim V_i) \alpha_i$. A simple object S_i has a dimension vector equal to α_i .

Let $\text{Ind } Q$ denote the set of isomorphism classes of indecomposable objects of $\text{mod } \mathbb{C}Q$. The theorem of Gabriel states that for each $\beta \in \Phi^+$, there is a unique class $[M] \in \text{Ind } Q$ with $\mathbf{d}_M = \beta$, and that all indecomposable objects of $\text{mod } \mathbb{C}Q$ are obtained this way. There is therefore a one-to-one correspondence between $\text{Ind } Q$ and Φ^+ , and we shall denote by $[\beta]$ the class $[M]$ in $\text{Ind } Q$ whose dimension vector is β .

The **Auslander-Reiten quiver** Γ_Q ([1] Chapter VII, [9] section 6) has as set of vertices $\text{Ind } Q$, and its arrows are irreducible morphisms between objects of $\text{Ind } Q$. It has a rigid mesh structure, as given in [9] Figure 13, page 49. The quiver Γ_Q is endowed with the translation τ ([1] page 225) which sends non-projective modules of $\text{Ind } Q$ onto non-injective modules of $\text{Ind } Q$. The translation τ stratifies Γ_Q into levels. The i^{th} level is the orbit under τ of the injective envelope of S_{i*} , where $*$ denotes the Dynkin diagram automorphism induced by w_0 . This level ends in the projective cover of S_i .

There is a natural order on the vertices of Γ_Q , given by $[\beta_1] \leq_Q [\beta_2]$ if and only if there is a path from $[\beta_1]$ to $[\beta_2]$ in Γ_Q . This induces a partial order \preceq_Q on Φ^+ by putting $\beta_1 \preceq_Q \beta_2$ whenever $[\beta_1] \leq_Q [\beta_2]$. The reflection ordering $\preceq_{\mathbf{w}_0}$ is then a linear refinement of \preceq_Q .

The path algebra $\mathbb{C}Q$ is an hereditary algebra. The Euler-Poincaré characteristic $\langle M_1, M_2 \rangle := \dim \text{Hom}(M_1, M_2) - \dim \text{Ext}^1(M_1, M_2)$ depends only on the dimension

vectors $\mathbf{d}_{M_1}, \mathbf{d}_{M_2}$ of M_1 and M_2 . One has $\langle [\beta_1], [\beta_2] \rangle = (\beta_1, \beta_2)_R$, where $(\cdot)_R$ is the **Ringel form** upon the Euclidean space $\mathbb{R}\Phi$. The matrix $R = (r_{i,j})$ of this form, in the basis of the simple roots, is given by

$$r_{i,j} := (\alpha_i, \alpha_j)_R = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i \longrightarrow j \text{ is in } Q, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.1 [21]

- i) Suppose $\beta_1 \preceq_Q \beta_2$ then $\dim \text{Ext}^1([\beta_1], [\beta_2]) = 0$ and therefore $\dim \text{Hom}([\beta_1], [\beta_2]) = (\beta_1, \beta_2)_R$.
- ii) Suppose $\beta_1 \succ_Q \beta_2$ then $\dim \text{Hom}([\beta_1], [\beta_2]) = 0$ and therefore $\dim \text{Ext}^1([\beta_1], [\beta_2]) = -(\beta_1, \beta_2)_R$.

Theorem 3.1 reduces the testing of condition (L) for a quiver Q , to computations in terms of Φ^+ .

We refer to Section 2 of [18] for a concise description of the Hall algebra construction of $U_q(\mathfrak{n}^+)$ and its link to PBW bases. The $\mathbb{C}(q)$ vector space with formal base vectors $\mathbf{u}_{[M]}$ indexed by isomorphism classes of $\text{mod } \mathbb{C}Q$, may be endowed with a product linked to the module structure. This defines the Hall algebra $\mathcal{H}(Q)$. Ringel's main theorem [20] states that sending the generators e_i to $\mathbf{u}_{[S_i]}$, establishes an isomorphism $\eta_Q : U_q(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{H}(Q)$.

Let $[M] = \bigoplus_{j=1}^N [\beta_j]^{\oplus t_j}$ be an isoclass with multiplicities of indecomposables given by $\mathbf{t}_M := (t_1, t_2, \dots, t_N)$. The PBW monomial $\mathbf{p}_{\mathbf{w}_0}(\mathbf{t}_M)$, with \mathbf{w}_0 adapted to Q is recovered, up to a multiplication by a well defined power of q , as the inverse image under η_Q of $\mathbf{u}_{[M]}$. Crystal operators in the Lusztig parametrization of \mathbf{w}_0 may therefore be seen as acting upon isomorphism classes of $\text{mod } \mathbb{C}Q$. One has $\tilde{e}_i[M_1] = [M_2]$ if and only if $\tilde{e}_i \mathbf{p}_{\mathbf{w}_0}(\mathbf{t}_{M_1}) = \mathbf{p}_{\mathbf{w}_0}(\mathbf{t}_{M_2}) \text{ mod } q^{-1}\mathcal{L}$.

Let us fix $i \in I$. The description of the action of \tilde{e}_i is given in terms of the set ([18] page 711) :

$$P_i(Q) := \{[X] \in \text{Ind}Q \mid \dim \text{Hom}(X, S_i) > 0\}.$$

The set $P_i(Q)$ has a poset structure $[X] \leq [Y]$ whenever there is a path from $[X]$ to $[Y]$ inside $P_i(Q)$. It is the same as the order induced by \leq_Q ([18] Proposition 4.3). Recall an **antichain** A of a $P_i(Q)$ is a set of mutually non-comparable elements. It defines the **order ideal** $J(A) := \{[X] \in P_i(Q) \mid \exists [Y] \in A, [X] \leq [Y]\}$. The correspondence $A \mapsto J(A)$ is one-to-one, and inclusion between order ideals induces a natural poset structure upon the set $\mathcal{A}_i(Q)$ of all antichains of $P_i(Q)$.

Given $A \in \mathcal{A}_i(Q)$, let C_A be the set of minimal elements of $P_i(Q) \setminus J(A)$, and define

$$\begin{aligned} [V_A] &:= \bigoplus_{[M] \in A} [M]; \\ [U_A] &:= \bigoplus_{[M] \in C_A} [\tau M]. \end{aligned}$$

The Lusztig move corresponding to A is then $\mathbf{l}_A := \mathbf{t}_{V_A} - \mathbf{t}_{U_A}$.

Each $A \in \mathcal{A}_i(Q)$ also defines a function $F_A : \mathbb{N}^N \rightarrow \mathbb{Z}$ given by

$$F_A(\mathbf{t}) := \sum_{X \in J(A)} \Delta_X(\mathbf{t})$$

where for $X = [\beta_k]$ with $\tau X = [\beta_{k'}]$, $\Delta_X(\mathbf{t}) = t_k - t_{k'}$ (with the convention that the second term is 0 if translation is not defined on X).

Theorem 3.2 ([18] Theorem 7.1)

Let $[M]$ be an isomorphism class of $\text{mod } \mathbb{C}Q$. Put $\zeta_i([M]) := \max_{A \in \mathcal{A}_i(Q)} F_A(\mathbf{t}_M)$. Then the subset $\{A \mid F_A(\mathbf{t}_M) = \zeta_i([M])\}$ of $\mathcal{A}_i(Q)$ admits a unique maximal element A_{\max} . There is an isoclass $[X]$ such that $[M] = [X] \oplus [U_{A_{\max}}]$, and the action of \tilde{e}_i is given then by $\tilde{e}_i[M] = [X] \oplus [V_{A_{\max}}]$.

In terms of Lusztig parameters, $\tilde{e}_i \mathbf{t}_M = \mathbf{t}_M + \mathbf{l}_{A_{\max}}$.

Corollary 3.3

Consider the Lusztig parametrization corresponding to \mathbf{w}_0 . The set of Lusztig moves of type i is given by

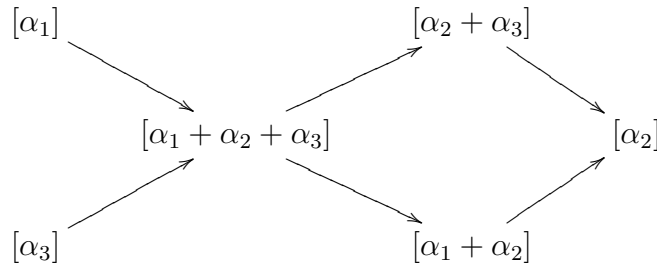
$$L_{\mathbf{w}_0}^{(i)} = \{\mathbf{l}_A \mid A \in \mathcal{A}_i(Q)\}.$$

Proof

In view of Theorem 3.2, given $A \in \mathcal{A}_i(Q)$, we need to exhibit a module on which \tilde{e}_i acts according to the move defined by A . Such a module is U_A . One has $\Delta_X(\mathbf{t}_{U_A}) \geq 0$ for any $[X] \in J(A)$, where as $\Delta_X(\mathbf{t}_{U_A}) = -1$ for any $[X] \in C_A$. These two properties ensure that for U_A , A_{\max} of Theorem 3.2 is A . \square

Example

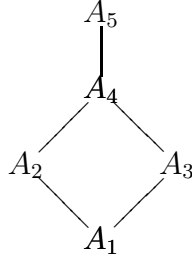
Let Q be the quiver $\cdot \xleftarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{3}$ of type A_3 , with adapted reduced expression $\mathbf{w}_0 = s_1 s_3 s_2 s_1 s_3 s_2$. The Auslander-Reiten quiver Γ_Q is



Let us study the action of \tilde{e}_2 . Using $(\cdot)_R$ and Theorem 3.1, one gets $P_2(Q) = \{[\alpha_2], [\alpha_1 + \alpha_2], [\alpha_2 + \alpha_3], [\alpha_1 + \alpha_2 + \alpha_3]\}$. $\mathcal{A}_2(Q)$ consists in 5 antichains :

Antichain A	U_A	V_A	\mathbf{l}_A
$A_1 = \{[\alpha_1 + \alpha_2 + \alpha_3]\}$	$[\alpha_1] \oplus [\alpha_3]$	$[\alpha_1 + \alpha_2 + \alpha_3]$	$(-1, -1, 1, 0, 0, 0)$
$A_2 = \{[\alpha_2 + \alpha_3]\}$	$[\alpha_3]$	$[\alpha_2 + \alpha_3]$	$(0, -1, 0, 1, 0, 0)$
$A_3 = \{[\alpha_1 + \alpha_2]\}$	$[\alpha_1]$	$[\alpha_1 + \alpha_2]$	$(-1, 0, 0, 0, 1, 0)$
$A_4 = \{[\alpha_1 + \alpha_2], [\alpha_2 + \alpha_3]\}$	$[\alpha_1 + \alpha_2 + \alpha_3]$	$[\alpha_1 + \alpha_2] \oplus [\alpha_2 + \alpha_3]$	$(0, 0, -1, 1, 1, 0)$
$A_5 = \{[\alpha_2]\}$	0	$[\alpha_2]$	$(0, 0, 0, 0, 0, 1)$

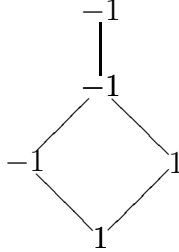
The set $\mathcal{A}_2(Q)$ has the following poset structure



Considering the structure of Γ_Q and the order ideals defined by members of $\mathcal{A}_2(Q)$, we get the F -functions data.

Antichain	$J(A)$	F_A
A_1	$\{[\alpha_1 + \alpha_2 + \alpha_3]\}$	t_3
A_2	$\{[\alpha_1 + \alpha_2 + \alpha_3], [\alpha_2 + \alpha_3]\}$	$t_3 + (t_4 - t_1)$
A_3	$\{[\alpha_1 + \alpha_2 + \alpha_3], [\alpha_1 + \alpha_2]\}$	$t_3 + (t_5 - t_2)$
A_4	$\{[\alpha_1 + \alpha_2 + \alpha_3], [\alpha_1 + \alpha_2], [\alpha_2 + \alpha_3]\}$	$t_3 + (t_4 - t_1) + (t_5 - t_2)$
A_5	$P_2(Q)$	$(t_4 - t_1) + (t_5 - t_2) + t_6$

Take $\mathbf{t}_M = (3, 2, 1, 1, 2, 0)$. One has $t_3 = 1$, $t_4 - t_2 = -2$, $t_5 - t_2 = 0$, $t_6 = 0$. Replacing each antichain A by the value of $F_A(\mathbf{t}_M)$ gives the following diagram



This pinpoints A_3 as A_{max} for $[M]$. Thus $\tilde{e}_2 \mathbf{t}_M = \mathbf{t}_M + \mathbf{l}_{A_3} = (2, 2, 1, 1, 3, 0)$.

4. Wiring diagrams and string cones

We shall restrict ourselves from now on to A_n type. The positive roots are $\beta = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$, $1 \leq i \leq j \leq n$. They are in a one-to-one correspondence with pairs (i, j) , $1 \leq i < j \leq n + 1$, β above being sent to $(i, j + 1)$. The fundamental representation $E(\omega_1)$ of \mathfrak{sl}_{n+1} has weights given by $\nu_j = -\omega_{j-1} + \omega_j$, $j = 1 \dots n + 1$ (with the convention that $\omega_0 = \omega_{n+1} = 0$). The weight ν_j may be seen as the weight of a one-box Young tableau \boxed{j} . The weights of fundamental representations $E(\omega_k)$, for $2 \leq k \leq n$ correspond to strictly increasing column tableaux of size k . The weight of a tableau is the sum of weights of its boxes. We shall therefore identify these weights with k -tuples $1 < j_1 < j_2 < \dots < j_k \leq n + 1$. The action of W on roots or on weights identifies with that of the symmetric group \mathfrak{S}_{n+1} on the respective multi-indices we considered.

The **wiring diagram** $\mathcal{WD}(\mathbf{w}_0)$ of a reduced expression $\mathbf{w}_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ consists in encoding \mathbf{w}_0 as an arrangement of pseudo-lines L_1, \dots, L_{n+1} drawn inside a vertical strip of \mathbb{R}^2 . The respective crossing points of these strands occur in levels according to the indices of \mathbf{w}_0 . Figure 4.1 gives a self-explaining example of the procedure in type A_3 .

Each strand L_i crosses another strand L_j once, and only once. The order of the strands L_i , $i = 1, \dots, n+1$ gets inverted while following $\mathcal{WD}(\mathbf{w}_0)$ from left to right. Sending the crossing $v_{i,j}$ ($i < j$) of lines L_i and L_j onto the couple (i, j) , establishes a one-to-one order preserving correspondence between these crossings enumerated from left to right and the reflection ordering $\preccurlyeq_{\mathbf{w}_0}$ of Φ^+ . If β_k is the k^{th} root in that order, with $i_k = i$, then the k^{th} crossing of $\mathcal{WD}(\mathbf{w}_0)$, that we shall denote by v_{β_k} , is on level i .

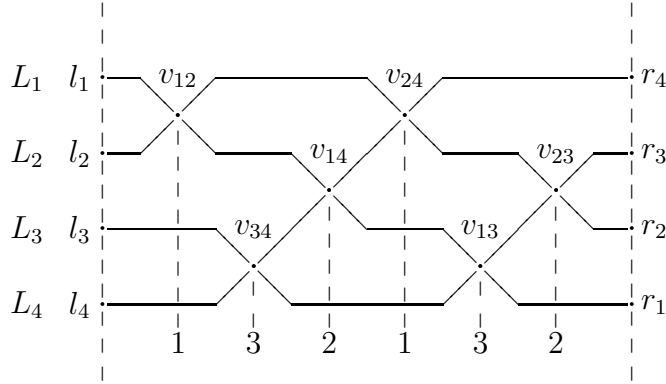


Figure 4.1 : wiring diagram of $\mathbf{w}_0 = s_1 s_3 s_2 s_1 s_3 s_2$ of type A_3 .

Let $G^\circ(\mathbf{w}_0)$ be the non-oriented graph obtained from $\mathcal{WD}(\mathbf{w}_0)$, whose vertices are the crossing points of pseudolines, and whose edges are given by pseudoline segments linking two crossing points. Likewise, let us denote by $G(\mathbf{w}_0)$ the non-oriented graph obtained in a similar way, by considering the vertices of $G^\circ(\mathbf{w}_0)$, as well as the vertices $l_1, \dots, l_{n+1}, r_1, \dots, r_{n+1}$ on the border of $\mathcal{WD}(\mathbf{w}_0)$.

Two vertices v_{β_k} and $v_{\beta_{k'}}$ with $k < k'$, $i_k = i$ and $i_{k'} = j$ are adjacent in $G^\circ(\mathbf{w}_0)$ if one of two cases occur. The adjacency is **diagonal** if one has $|i - j| = 1$ and $i_l \neq i, i_l \neq j$ for any l satisfying $k < l < k'$. The adjacency is **horizontal** when $i = j$ and either $i_l < i - 1$ for all $k < l < k'$, or $i_l > i + 1$ for all $k < l < k'$. We see in Figure 4.1 above that v_{12} and v_{14} are diagonally adjacent, whereas v_{12} and v_{24} are horizontally adjacent.

The wiring diagram $\mathcal{WD}(\mathbf{w}_0)$ defines a set of bounded chambers. Such a chamber may be indexed by set of indices of the pseudo-lines passing above it. This set may be seen as indices of a column Young tableau. One gets a one-to-one correspondence between these chambers and a set of weights inside $\bigcup_{k=1}^n W\omega_k$.

A vertex v_β of $G^\circ(\mathbf{w}_0)$ may be assigned either the weight $\lambda^-(v_\beta)$ of the chamber left to it, or the weight $\lambda^+(v_\beta)$ of the chamber right to it. It is well known that if $\beta = s_{i_1} s_{i_2} \dots s_{i_{k-1}}(\alpha_{i_k})$, then

$$\lambda^-(\beta) = s_{i_1} s_{i_2} \dots s_{i_{k-1}}(\omega_{i_k}), \quad \lambda^+(\beta) = s_{i_1} s_{i_2} \dots s_{i_{k-1}} s_{i_k}(\omega_{i_k}).$$

Figure 4.2 below gives the example of the chamber system for $\mathbf{w}_0 = s_1 s_3 s_2 s_1 s_3 s_2$ of type A_3 . The vertex v_{14} is between the chamber labelled 12 to the left, and that labelled 24 to the right. One has $\lambda^-(v_{14}) = \nu_1 + \nu_2$, $\lambda^+(v_{14}) = \nu_2 + \nu_4$.

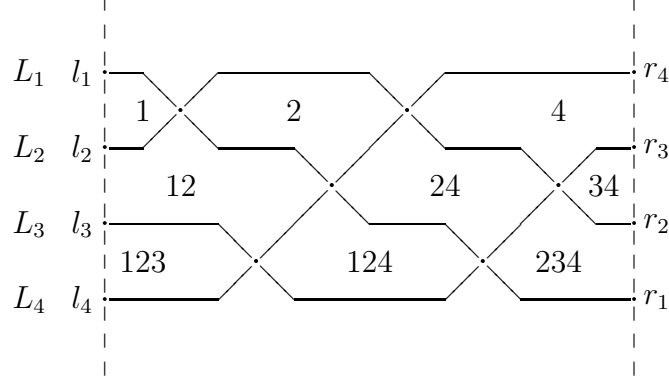


Figure 4.2 : chamber system of $\mathcal{WD}(\mathbf{w}_0)$, for $\mathbf{w}_0 = s_1 s_3 s_2 s_1 s_3 s_2$ of type A_3 .

Gleizer-Postnikov ([10] section 5) obtain a system of defining inequalities for the string cone $\mathcal{C}_{\mathbf{w}_0}$, by transforming for every $i \in I$, the graph $G(\mathbf{w}_0)$ into an oriented graph $G(\mathbf{w}_0, i)$. The pseudolines L_1, L_2, \dots, L_i are oriented backwards, and the pseudolines L_{i+1}, \dots, L_{n+1} forwards. The resulting graph is acyclic.

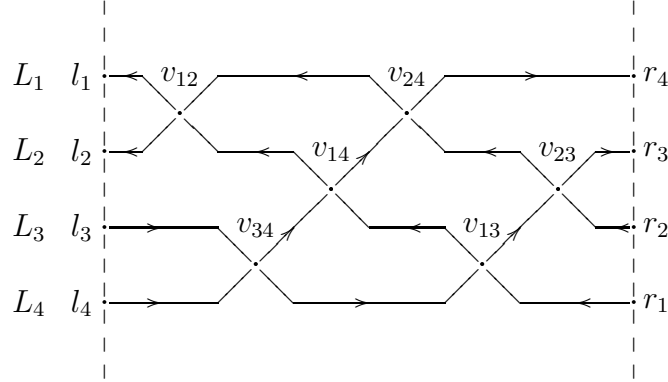


Figure 4.3 : $G(\mathbf{w}_0, 2)$, $\mathbf{w}_0 = s_1 s_3 s_2 s_1 s_3 s_2$, type A_3

Let π be a path inside $G(\mathbf{w}_0, i)$. We shall qualify the two configurations below as forbidden crossings :



(path π in plain line)

A **Gleizer-Postnikov path (GP-path) of type i** is a path of $G(\mathbf{w}_0, i)$, starting from l_{i+1} on the left border, ending at l_i , and that does not contain forbidden crossings.

An example of such a path in Figure 4.3 above is $l_3 \longrightarrow v_{34} \longrightarrow v_{13} \longrightarrow v_{14} \longrightarrow v_{24} \longrightarrow v_{12} \longrightarrow l_2$. There are 5 such paths inside $G(\mathbf{w}_0, 2)$.

Remark : Gleizer-Postnikov paths were called rigorous paths in [10]. We have translated the vertical setting of [10] to an horizontal one, which is more natural when comparing wiring diagrams to Auslander-Reiten quivers.

Let π be a GP-path. If π enters a vertex v_{β_j} following the line L_h , and leaves it following L_l , assign to it the value

$$k_j := \begin{cases} 1 & \text{if } h > l \\ -1 & \text{if } h < l \\ 0 & \text{if } h = l \end{cases}$$

If v_{β_j} is not a vertex of π , put $k_j := 0$. The coordinates k_j , $j = 1, \dots, N$ define a vector \mathbf{k}_π of \mathbb{Z}^N .

Take as an example the GP-path above, $\pi = l_3 \longrightarrow v_{34} \longrightarrow v_{13} \longrightarrow v_{14} \longrightarrow v_{24} \longrightarrow v_{12} \longrightarrow l_2$. The path π starts on strand L_3 , changes to strand L_1 at v_{13} (hence a positive contribution), then passes from strand L_1 to strand L_4 at v_{14} (hence a negative contribution), and finally its last change of strands occurs at v_{24} , where π passes from L_4 to L_2 (hence a positive contribution). The vertices v_{13} , v_{14} , v_{24} occur respectively in positions 5, 3 and 4 while we follow $\mathcal{WD}(s_1 s_3 s_2 s_1 s_3 s_2)$ from left to right. We get $\mathbf{k}_\pi = (0, 0, -1, 1, 1, 0)$.

As $G(\mathbf{w}_0, i)$ is acyclic, there are only finitely many different Gleizer-Postnikov paths of type i .

Theorem 4.1 ([10] corollary 5.8)

Let $K_{\mathbf{w}_0}^{GP}$ be the set of all vectors \mathbf{k}_π , where π varies over all possible GP-paths of all possible types $i \in I$. Then $K_{\mathbf{w}_0}^{GP}$ defines the string cone $\mathcal{C}_{\mathbf{w}_0}$. One has

$$\mathcal{S}_{\mathbf{w}_0} = \{\mathbf{t} \in \mathbb{N}^N \mid \forall \mathbf{k}_\pi \in K_{\mathbf{w}_0}^{GP}, \mathbf{k}_\pi \cdot \mathbf{t} \geq 0\}.$$

Fix $i \in I$, and consider $G(\mathbf{w}_0, i)$. Let us denote by $\delta_i^>$ the segment of L_{i+1} starting from the left border on l_{i+1} , up to its intersection v_{α_i} with L_i . In a similar way, we denote by $\delta_i^<$ the segment of L_i starting from v_{α_i} and going back to l_i on the left border following L_i . The concatenation $\delta_i := \delta_i^> * \delta_i^<$ is then a path starting from l_{i+1} and ending in l_i . We shall call it the **limiting path of type i** .

Observe δ_i serves as the boundary of a set of chambers $\mathcal{Z}_i(\mathbf{w}_0)$. These chambers are those lying below L_i , above L_{i+1} and left of v_{α_i} . Let us denote by $Z_i(\mathbf{w}_0)$ the set of vertices of $G^\circ(\mathbf{w}_0)$ which are the rightmost vertices of chambers of $\mathcal{Z}_i(\mathbf{w}_0)$.

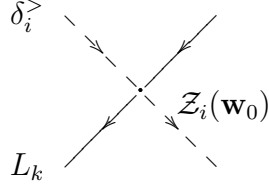
Lemma 4.2

Consider v a vertex of δ_i other than l_i , l_{i+1} , v_{α_i} . Suppose this vertex is a crossing of δ_i with a line L_k ($k \neq i, i+1$). We have then the two following cases :

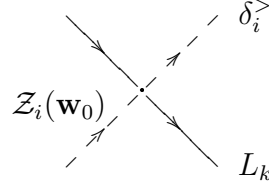
- i) If $v \in \delta_i^>$, then L_k crosses δ_i going inside $\mathcal{Z}_i(\mathbf{w}_0)$.
- ii) If $v \in \delta_i^<$, then L_k crosses δ_i going outside $\mathcal{Z}_i(\mathbf{w}_0)$.

Proof of lemma 4.2 :

i) We have to eliminate the possibility of L_k going out of $\mathcal{Z}_i(\mathbf{w}_0)$ while crossing $\delta_i^>$. By definition, $\mathcal{Z}_i(\mathbf{w}_0)$ lies above $\delta_i^>$, hence L_k crosses $\delta_i^>$ going downwards. Depending on the orientation of L_k , we obtain the two following cases :



a) $k < i$



b) $k > i + 1$

In case a), L_k would have to recross L_{i+1} in order to return to l_k which lies above l_{i+1} on the left border. In case b), l_k lies below l_{i+1} on the left border, so L_k would have to cross L_{i+1} a first time in order to reach the crossing point in b) from above. Both cases are impossible since they would imply at least two crossings of L_k and L_{i+1} in $\mathcal{WD}(\mathbf{w}_0)$.

Statement ii) is proved by symmetric arguments. \square

Corollary 4.3

δ_i is a GP-path.

Proof :

Clearly, v_{α_i} is not a forbidden crossing. The other vertices of δ_i belong to the cases detailed in the lemma above, none of them being forbidden. \square

Proposition 4.4

Let π be a GP-path of type i . Then π stays inside $\mathcal{Z}_i(\mathbf{w}_0)$.

Proof :

The path π starts at l_{i+1} and ends at l_i which are inside $\mathcal{Z}_i(\mathbf{w}_0)$. Suppose it exits $\mathcal{Z}_i(\mathbf{w}_0)$ at some vertex $v_1 \in \delta_i$. By the lemma above, one must have $v_1 \in \delta_i^<$. The same lemma shows π must return inside $\mathcal{Z}_i(\mathbf{w}_0)$ through a crossing point $v_2 \in \delta_i^>$. Now the segment of δ_i between v_1 and v_2 goes from v_2 to v_1 , so we can create a cycle. This is in contradiction with the fact that $G(\mathbf{w}_0, i)$ is acyclic. \square

5. Hammocks

We continue to restrict ourselves to A_n case, with Q and \mathbf{w}_0 fixed. Let us denote by $[\beta : \alpha_i]$ the coefficient of α_i in the expression of β . The **hammock** of type i , $i \in I$ ([6]) is $H_i(Q) := \{[\beta] \mid [\beta : \alpha_i] > 0\}$. The set $P_i(Q)$ is a subset of $H_i(Q)$, and we shall see the combinatorics of Lusztig's moves of type i is obtained from that of $H_i(Q)$. The structure of $H_i(Q)$ itself is very simple, and is deduced from the Coxeter element c attached to Q .

Recall [5] that one may renumber the vertices of Q by a permutation i_1, i_2, \dots, i_n of I , such that for every arrow $j \rightarrow k$ of Q , one has $i_j < i_k$. The Coxeter element is then given by $c = s_{i_1} s_{i_2} \dots s_{i_n}$. The action of c on Φ^+ is the mirror image of the action of the translation τ upon Γ_Q : if $N = \tau M$ then $\mathbf{d}_N = c\mathbf{d}_M$.

In type A_n , as $W \cong \mathfrak{S}_{n+1}$, c is an $n + 1$ -cycle. Its expression may be constructed out of Q by the following algorithm ([22] Lemma 4.2) :

- Start with just the element $n + 1$.
- Proceed in decreasing order $i = n, n - 1, \dots, 2$:
 - If inside Q , one has $\overset{i-1}{\cdot} \longleftarrow \overset{i}{\cdot}$, add i to the right of the indices already written.
 - If inside Q , one has $\overset{i-1}{\cdot} \longrightarrow \overset{i}{\cdot}$, add i to the left of the indices already written.
- Finish by adding 1 to the left of the n indices already written.

The sequence of indices $j_1 j_2 \dots j_{n+1}$ thus obtained is a $n + 1$ -cycle expression of c . The consequence of this specific algorithm is that the $n + 1$ -cycle expressions of c verify a special "segment" property.

Lemma 5.1 ([22], Proposition 4.3)

For every $i \in \{1, \dots, n\}$, there is a cycle expression $c = (j_1, \dots, j_i, j_{i+1} \dots j_{n+1})$ where :

- j_1, j_2, \dots, j_i is a permutation of $1, \dots, i$.
- $j_{i+1}, j_{i+2}, \dots, j_{n+1}$ is a permutation of $i + 1, i + 2, \dots, n + 1$.

We shall refer to this cycle expression as the **i -segmented expression** of c , and denote it by $(j_1, \dots, j_i \mid j_{i+1}, \dots, j_{n+1})$.

Example

Consider the quiver $Q : \overset{1}{\cdot} \longleftarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot} \longleftarrow \overset{4}{\cdot}$ of type A_4 . One has $c = s_2 s_1 s_4 s_3$. The algorithm above gives

$$5 \xrightarrow{\text{right}} 54 \xrightarrow{\text{left}} 354 \xrightarrow{\text{right}} 3542 \xrightarrow{\text{left}} 13542.$$

One may verify the 5-cycle expression (13542) obtained agrees with c . The i -segmented expressions of c are then respectively

$$\begin{array}{ll} i = 1 : (1 \mid 3542) & i = 3 : (213 \mid 54) \\ i = 2 : (21 \mid 354) & i = 4 : (4213 \mid 5) \end{array}$$

The positions of $i, i + 1$ in the i -segmented expression of c depend on the neighbourhood of i in Q :

$$\begin{array}{ll} \overset{i-1}{\cdot} \longrightarrow \overset{i}{\cdot} \text{ in } Q : j_i = i & \overset{i-1}{\cdot} \longleftarrow \overset{i}{\cdot} \text{ in } Q : j_1 = i \\ \overset{i}{\cdot} \longrightarrow \overset{i+1}{\cdot} \text{ in } Q : j_{i+1} = i + 1 & \overset{i}{\cdot} \longleftarrow \overset{i+1}{\cdot} \text{ in } Q : j_{n+1} = i + 1 \end{array}$$

In the example above, the neighbourhood of vertex 2 in Q is $\overset{1}{\cdot} \longleftarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot}$, so that 2 appears in first position of the segment 21, and 3 appears in first position of the segment 354.

Given an integer $m \geq 1$, we shall denote by $[m]$ the set $\{1, 2, \dots, m\}$.

Proposition 5.2 *Let Q be a quiver of type A_n .*

- a) *The set $H_i(Q)$ for $i \in I$, seen as a subgraph of Γ_Q , has a structure isomorphic to $[i] \times [n+1-i]$ as given in Figure 5.1, with $[\beta_{min}]$ the isoclass of the projective cover of the simple module S_i and $[\beta_{max}]$ the isoclass of the injective envelope of S_i .*
- b) *If $(j_1 j_2 \dots j_i \mid j_{i+1} \dots j_{n+1})$ is the i -segmented writing of c , then the vertex at position (k, l) in the Figure 5.1 is $[\alpha_{j_k} + \alpha_{j_{k+1}} + \dots + \alpha_{j_l-1}]$.*

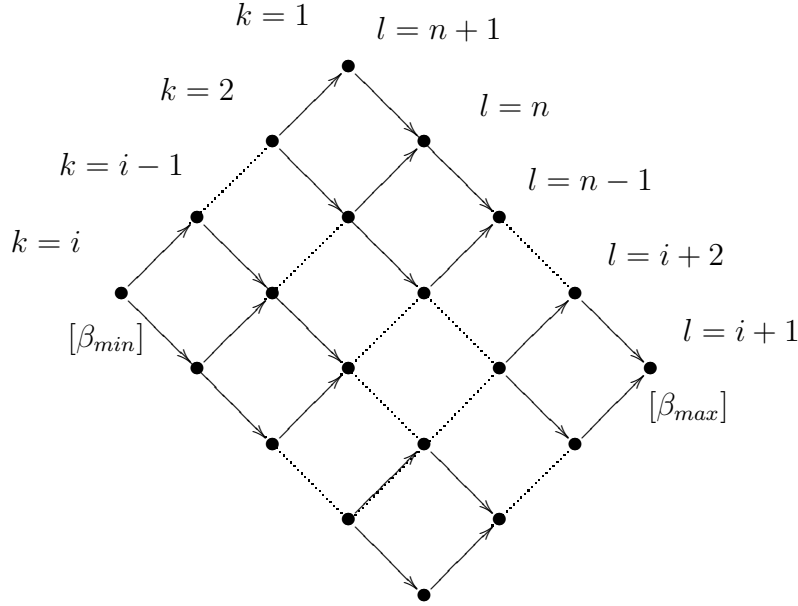


Figure 5.1 : Hammock $H_i(Q)$ of type A_n

The proposition consists in a computation well known to specialists. The reader may consult [9] section 6.5 for technical details, especially pages 52-54 (our Figure 5.1 corresponds to the first scheme in [9] Figure 15). We provide here some guidelines for non-specialists.

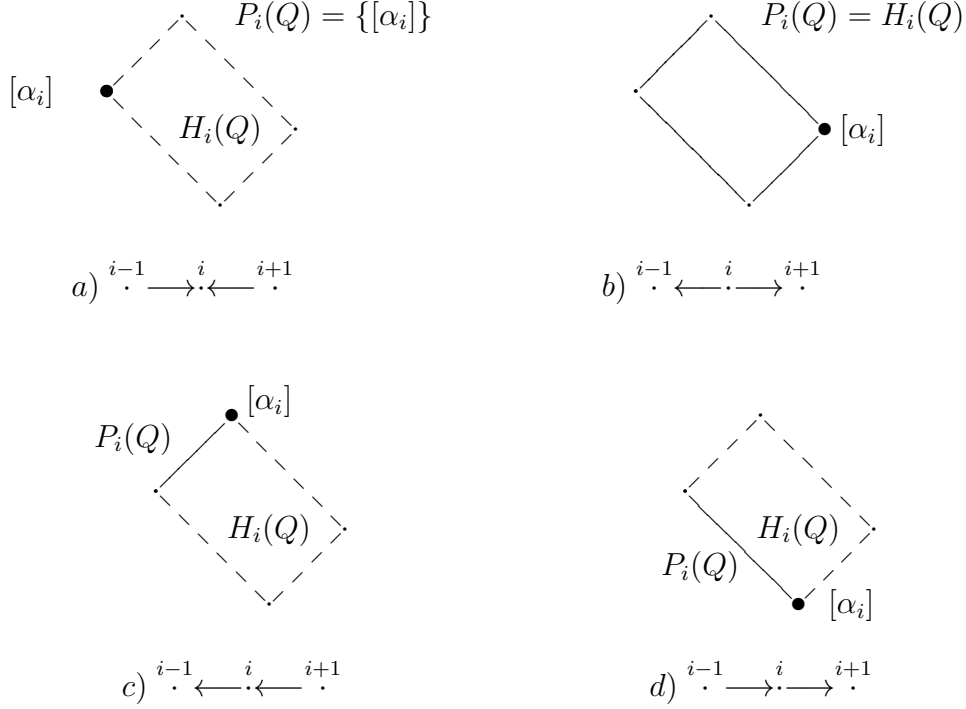
The set $H_i(Q)$ is particularly simple to compute in the case of the " i -regular" quiver $1 \xrightarrow{1} 2 \xrightarrow{2} \dots \xrightarrow{i} i \xleftarrow{i} \dots \xleftarrow{n-1} n \xleftarrow{n}$ admitting i as its unique sink. The modules at positions $(1, n+1), (2, n+1), \dots, (i, n+1), (i, n), \dots, (i, i+1)$ in Figure 5.1 are the respective projective covers of the simple modules S_1, \dots, S_n . Their dimension vectors are directly obtained out of Q .

The mesh structure of Γ_Q [9], verifies additivity for dimension vectors. For each "square" configuration of vertices $[X], [Z_1], [Z_2], [Y]$, respectively at positions $(k, l), (k-1, l), (k, l-1), (k-1, l-1)$ in Figure 5.1 one has $\mathbf{d}_X + \mathbf{d}_Y = \mathbf{d}_{Z_1} + \mathbf{d}_{Z_2}$. Thus the knowledge of the dimension vectors for the "slice" of projective modules allows to compute the rest of $H_i(Q)$.

An arbitrary quiver Q may be obtained from the i -regular one by a sequence of orientations changes transforming a source vertex j , with $j \neq i$, into a sink. The

effect of this transformation on vertices of Γ_Q other than $[\alpha_j]$ consists in applying the corresponding BGP-reflection functor Σ_j [5]. This is the case of the vertices of $H_i(Q)$, which never contains $[\alpha_j]$. In terms of dimension vectors, $\mathbf{d}_{\Sigma_j X} = s_j \mathbf{d}_X$. One checks this change is coherent with the change in the Coxeter element c due to the change of orientation.

One may verify by direct computation that $P_i(Q)$ is the order ideal defined by $[\alpha_i]$ inside $H_i(Q)$. The possible cases, according to the neighbourhood of i inside Q , are :



The correspondence $\Psi : \Gamma_Q \longrightarrow G^\circ(\mathbf{w}_0)$, $[\beta] \mapsto v_\beta$ will allow us to transfer the combinatorial results above to the setting of $\mathcal{WD}(\mathbf{w}_0)$. In particular, the translation operation τ on Γ_Q corresponds under Ψ , to an operation τ_{wd} on vertices of $G^\circ(\mathbf{w}_0)$. It is well known ([2], Lemma 2.11), that for every $i \in I$, Ψ establishes an order preserving one-to-one correspondence between the i -th translation level of Γ_Q and crossings on the i^{th} level of $\mathcal{WD}(\mathbf{w}_0)$. Given k between 1 and N , let k^- denote the maximal index j such that $j < k$ and $s_{i_j} = s_{i_k}$ in \mathbf{w}_0 (if such an index exists). The isoclass $[\beta_k]$ is not projective exactly when k^- is defined. One has then $\tau_{wd}(v_{\beta_k}) = v_{\beta_{k^-}} = v_{c\beta_k}$.

Let us define $H_i(\mathbf{w}_0) := \Psi(H_i(Q))$.

Proposition 5.3

Fix $i \in I$ and consider $H_i(Q)$ and $H_i(\mathbf{w}_0)$ respectively as subgraphs of Γ_Q and $G^\circ(\mathbf{w}_0)$. Then Ψ restricted to $H_i(Q)$ establishes an isomorphism of non-oriented graphs between $H_i(Q)$ and $H_i(\mathbf{w}_0)$.

Proof :

Let $[\beta_k] \longrightarrow [\beta_{k'}]$, $k < k'$ be an arrow of $H_i(Q)$. By [22] Proposition 1.2, the vertices i_k and $i_{k'}$ are linked in D , and there is no occurrence either of a reflection $s_{i_j} = s_{i_k}$ or of a reflection $s_{i_j} = s_{i_{k'}}$ in the positions j between $k+1$ and $k'-1$. This means that v_{β_k} and $v_{\beta_{k'}}$ are diagonally adjacent inside $H_i(\mathbf{w}_0)$.

Suppose now v_{β_k} and $v_{\beta_{k'}}$, $k < k'$, are adjacent vertices of $H_i(\mathbf{w}_0)$. A reduced expression \mathbf{w}_0 adapted to a quiver is alternating ([22] Lemma 1.4), in the sense that if j is linked to l in D , then between two occurrences of a reflection s_j , there is one and only one occurrence of a reflection s_l . This excludes the possibility of horizontal adjacencies inside $H_i(\mathbf{w}_0)$. We have therefore $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$, $\beta_{k'} = s_{i_1} \dots s_{i_{k'-1}}(\alpha_{i_{k'}})$, with $(\alpha_{i_k}, \alpha_{i_{k'}}) = -1$. One has $(\beta_k, \beta_{k'}) = (\alpha_{i_k}, s_{i_k} s_{i_{k+1}} \dots s_{i_{k'-1}}(\alpha_{i_{k'}}))$ by invariance of the Cartan scalar product under action of W . Now $i_{k+1}, \dots, i_{k'-1}$ are all different from i_k and $i_{k'}$, which implies that $(\alpha_{i_k}, s_{i_k} s_{i_{k+1}} \dots s_{i_{k'-1}}(\alpha_{i_{k'}})) = 1$.

By [2] Lemma 2.11, $(\beta_k, \beta_{k'}) = 1$ means there is a path inside Γ_Q from $[\beta_k]$ to $[\beta_{k'}]$. Now this path may have only one vertex on each of the adjacent levels i_k and $i'_{k'}$, so it can cross them only once. In view of the mesh structure of Γ_Q , the only possibility is that this path is reduced to a single arrow $[\beta_k] \longrightarrow [\beta_{k'}]$. \square

Proposition 5.4

One has $Z_i(\mathbf{w}_0) = \Psi(P_i(Q))$. Furthermore, these sets, seen as non-oriented subgraphs respectively of Γ_Q and $G^\circ(\mathbf{w}_0)$, are isomorphic.

Only the first part of the proposition needs to be proved, the second then follows, using Proposition 5.3. We shall need two results concerning the matrix $R = (r_{i,j})$ of the Ringel form of Q . Let us define for $i \in I$, $\rho_i := \sum_{k=1}^n r_{k,i} \omega_k$. The weight ρ_i is such that its coordinates in the basis of fundamental weights are given by the i -th column of R . In a similar manner, let us put, corresponding to the i -th line of R , $\rho_i^t := \sum_{k=1}^n r_{i,k} \omega_k$.

Lemma 5.5

The action of the Coxeter element on the vectors ρ_i , $i \in I$ is given by $c\rho_i = -\rho_i^t$.

Proof :

Case by case analysis, following the 4 cases for the neighbourhood of i in Q . Our convention is $\omega_0 = \omega_{n+1} = 0$.

Case 1 : i is a source of Q

We have in this case $\rho_i = \omega_i$, $\rho_i^t = -\omega_{i-1} + \omega_i - \omega_{i+1}$ and $c = s_i c_1$, c_1 being a product of simple reflections s_j with $j \neq i$. As $s_j(\omega_i) = \omega_i$ for $j \neq i$, we see $c\rho_i = s_i(\omega_i) = \omega_{i-1} - \omega_i + \omega_{i+1}$.

Case 2 : The neighbourhood of i in Q is of the form $\overset{i-1}{\cdot} \longrightarrow \overset{i}{\cdot} \longrightarrow \overset{i+1}{\cdot}$

We have in this case $\rho_i = -\omega_{i-1} + \omega_i$, $\rho_i^t = \omega_i - \omega_{i+1}$ and $c = c_1 s_i c_2$, where c_1 is a product of simple reflections s_j with $j \geq i+1$, and c_2 a product of simple reflections s_j with $j \leq i-1$. We get

$$\begin{aligned} c_1 \rho_i &= \rho_i, \\ s_i c_1 \rho_i &= s_i \rho_i = -\omega_i + \omega_{i+1} \\ c_2 s_i c_1 \rho_i &= -\omega_i + \omega_{i+1}. \end{aligned}$$

The other two cases follow by similar computations. \square

Recall we associated to a vertex $v_\beta \in G^\circ(\mathbf{w}_0)$, the weight $\lambda^+(v_\beta)$. Let φ_R be the linear mapping of the Euclidean space $\mathbb{R}\Phi$ defined by $\varphi_R(\alpha_i) = -\rho_i$, $i \in I$.

Theorem 5.6 ([22] Theorem 2.4)

$$\forall \beta \in \Phi^+, \quad \lambda^+(v_\beta) = \varphi_R(\beta).$$

The key of the theorem is [22] Lemma 2.2 : if k is a sink of an ADE quiver Q , $Q' := s_k Q$, and R, R' are the respective Ringel matrices, then $s_k \varphi_{R'} = \varphi_R s_k$. This lemma implies φ_R commutes with c^{-1} , and hence with $c = (c^{-1})^n$. If $\beta \in \Phi^+$ is such that $c\beta \in \Phi^+$ (that is, if τ is defined on $[\beta]$), then $c\lambda^+(v_\beta) = c\varphi_R(\beta) = \varphi_R(c\beta)$. We have then that $c\lambda^+(v_\beta) = \lambda^+(v_{c\beta}) = \lambda^-(v_\beta)$.

Proof of Proposition 5.4

For any $\beta \in \Phi^+$, one has that $(\beta, \alpha_i)_R = (\beta, \rho_i)$. This implies, by Theorem 3.1, that $[\beta] \in P_i(Q)$ if and only if $(\beta, \rho_i) > 0$.

Let us apply a similar analysis for a vertex v_β of $Z_i(\mathbf{w}_0)$. A chamber is in $\mathcal{Z}_i(\mathbf{w}_0)$ if the strand L_i passes above it, and the strand L_{i+1} passes below it. This means the index i appears in the labelling of the chamber, and that the index $i+1$ does not appear. Now the index i corresponds to the weight $-\omega_{i-1} + \omega_i$ and the index $i+1$ to the weight $-\omega_i + \omega_{i+1}$. The condition above in terms of strands, translates in terms of weights, to having ω_i with coefficient 1 in $\lambda^-(v_\beta)$. Thus $v_\beta \in Z_i(\mathbf{w}_0)$ if and only if $(\alpha_i, \lambda^-(v_\beta)) > 0$.

There is only one projective indecomposable module in $P_i(Q)$, namely the projective cover of S_i , which is on the i -th translation level of Γ_Q . Likewise, the left border chamber of $\mathcal{WD}(\mathbf{w}_0)$ on level j , $j \in I$, has the weight ω_j . Hence the only vertex on the left border of $G^0(\mathbf{w}_0)$ verifying $(\alpha_i, \lambda^-(v_\beta)) > 0$, is the one on the i^{th} level.

It remains to verify Proposition 5.4 for non-projective vertices of Γ_Q . Lemma 5.5 gives us, for all i, k that $(\alpha_k, \rho_i) = -(\alpha_i, c\rho_k)$. If $\beta = \sum_{k=1}^n m_k \alpha_k$, then

$$\begin{aligned} (\beta, \rho_i) &= \sum_{k=1}^n m_k (\alpha_k, \rho_i) \\ &= -\sum_{k=1}^n m_k (\alpha_i, c\rho_k) \\ &= (\alpha_i, c(-\sum_{k=1}^n m_k \rho_k)) \\ &= (\alpha_i, c\lambda^+(v_\beta)) \\ &= (\alpha_i, \lambda^-(v_\beta)). \end{aligned}$$

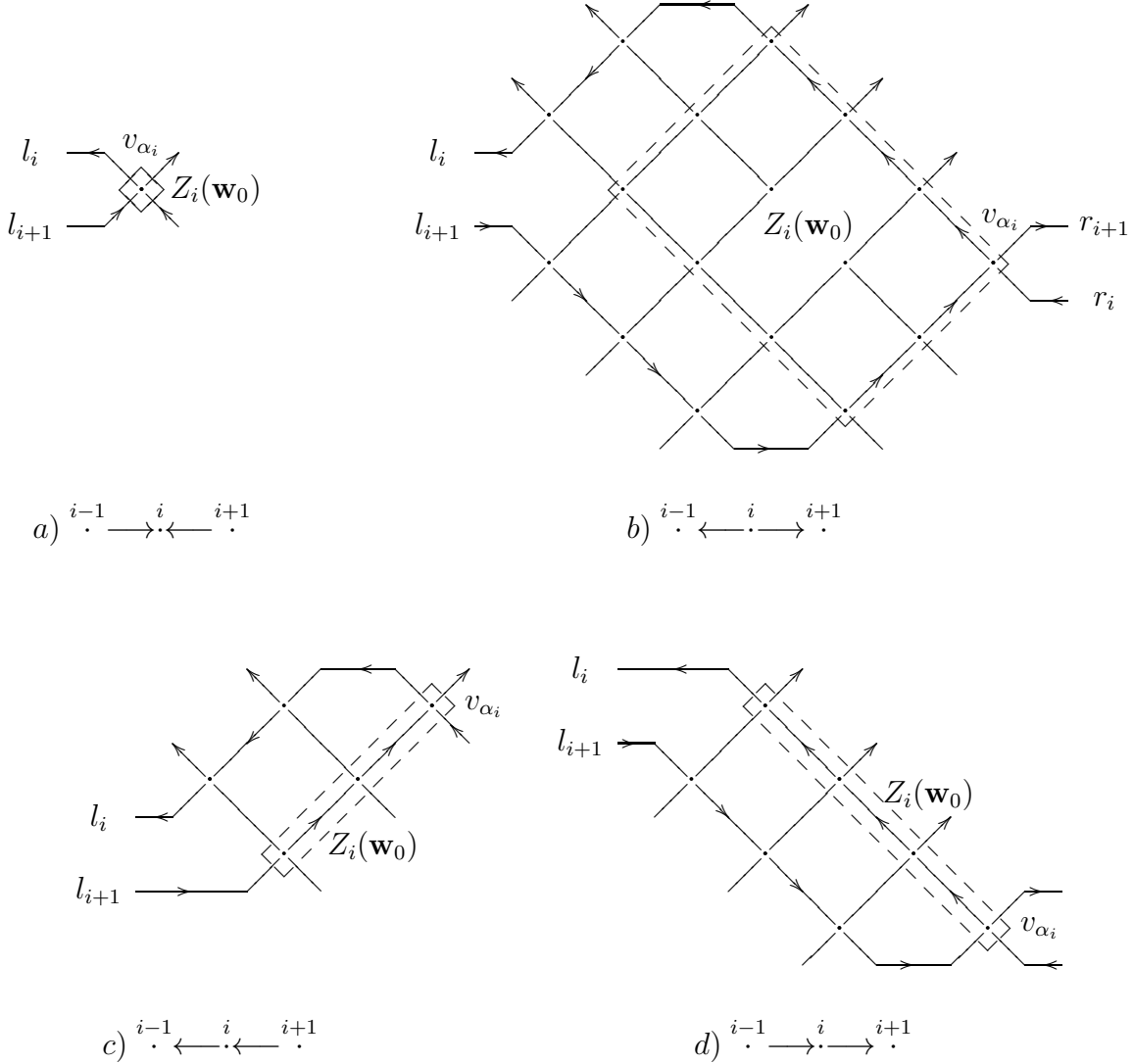
We see that $(\beta, \rho_i) > 0$ if and only if $(\alpha_i, \lambda^-(v_\beta)) > 0$ so the first statement of the proposition is verified. \square

6. String cone inequalities and Lusztig moves

A vertex occurring in a GP-path of type i must belong to a chamber of $\mathcal{Z}_i(\mathbf{w}_0)$. We shall denote by $Y_i(\mathbf{w}_0)$ the set of these vertices. We have already accounted for the set $Z_i(\mathbf{w}_0)$ which consists of right-most vertices of the chambers in $\mathcal{Z}_i(\mathbf{w}_0)$. It remains to compute $Y_i(\mathbf{w}_0) \setminus Z_i(\mathbf{w}_0)$. As we shall now see, these vertices lie on the border path δ_i .

Proposition 6.1

Fix $i \in I$. Then the subgraph with vertices $Y_i(\mathbf{w}_0)$ is given inside $G(\mathbf{w}_0, i)$, according to the neighbourhood of i inside Q , by Figures 6.1 below (with the relative position of $Z_i(\mathbf{w}_0)$ inside $Y_i(\mathbf{w}_0)$ delimited by segmented lines) :



Figures 6.1 : structure of $Y_i(\mathbf{w}_0)$ inside $G(\mathbf{w}_0, i)$.

Proof :

The key to the proof is the combinatorics of $H_i(Q)$ (and therefore of $H_i(\mathbf{w}_0)$). Let $(j_1, j_2 \dots j_i \mid j_{i+1} \dots j_{n+1})$ be the i -th segmented expression of c . The vertex of $H_i(Q)$ at position (k, l) in Figure 5.1, is mapped by Ψ onto the intersection of the pseudo-lines L_{j_k} and L_{j_l} , where $j_k \leq i$ and $j_l \geq i+1$. The structure of $Z_i(\mathbf{w}_0)$ inside $G(\mathbf{w}_0, i)$ follows then from Proposition 5.4. It remains to understand the position of the border path δ_i with respect to $Z_i(\mathbf{w}_0)$.

We shall call the set of vertices $v_\beta \in Z_i(\mathbf{w}_0)$ such that either $\tau_{wd}(v_\beta)$ is not defined, or $\tau_{wd}(v_\beta) \notin Z_i(\mathbf{w}_0)$, the left border of $Z_i(\mathbf{w}_0)$. We shall call the set of vertices $v_\beta \in Z_i(\mathbf{w}_0)$

which are not the image under τ_{wd} of another vertex of $Z_i(\mathbf{w}_0)$, the right border of $Z_i(\mathbf{w}_0)$. We shall see the limiting path δ_i consists of vertices either lying on the right border of $Z_i(\mathbf{w}_0)$ or of vertices obtained by applying the translation τ_{wd} on a vertex lying on the left border of $Z_i(\mathbf{w}_0)$. This ensures that vertices of $Y_i(\mathbf{w}_0)$ which do not belong to $Z_i(\mathbf{w}_0)$, lie on δ_i .

Let us consider the part $\delta_i^<$ of δ_i lying on the pseudoline L_i .

Case 1 : One has $\overset{i-1}{\cdot} \xrightarrow{\cdot} \overset{i}{\cdot}$ inside Q .

The i -segmented expression of c is such that $j_i = i$. In view of Proposition 5.2 b), L_i exits $Z_i(\mathbf{w}_0)$ on the leftmost vertex $v_{\beta_{min}}$ of $H_i(\mathbf{w}_0)$. Now $[\beta_{min}]$ is a projective isoclass of $\text{mod } \mathbb{C}Q$, hence $v_{\beta_{min}}$ is on the left border of $G^\circ(\mathbf{w}_0)$. The pseudoline L_i goes directly to l_i after leaving this vertex. The segment $\delta_i^<$ contains only one vertex of $Z_i(\mathbf{w}_0)$, namely $v_{\beta_{min}}$, which lies on the right border of this set.

Case 2 : One has $\overset{i-1}{\cdot} \xleftarrow{\cdot} \overset{i}{\cdot}$ inside Q .

This time $j_1 = i$, so L_i leaves $Z_i(\mathbf{w}_0)$ on level 1. Recall j_{i+1}, \dots, j_{n+1} is a permutation of the interval $i+1, i+2, \dots, n+1$, so $Z_i(\mathbf{w}_0)$ contains all vertices $L_i \cap L_k$, $k \geq i+1$, these vertices being on the right border of $Z_i(\mathbf{w}_0)$.

Let us fix $k \leq i-1$, and apply τ_{wd} on the leftmost vertex $L_{j_k} \cap L_{j_{n+1}}$ of $Z_i(\mathbf{w}_0)$ on level k . The result is, according to the i -segmented expression of c , the vertex $L_{j_{k+1}} \cap L_{j_1}$. Yet $j_1 = i$, and j_2, j_3, \dots, j_i is a permutation of $1, 2, \dots, i-1$. We see all vertices $L_i \cap L_k$ with $k \leq i-1$ lie on $\delta_i^<$. We have accounted for all vertices lying on L_i , so that after leaving $Y_i(\mathbf{w}_0)$, L_i goes to l_i .

The case of the part $\delta_i^>$ of δ_i lying on strand L_{i+1} is totally symmetric, so we won't give all details. The two cases to consider, correspond to the orientations of the arrow between i and $i+1$. If one has $\overset{i}{\cdot} \xleftarrow{\cdot} \overset{i+1}{\cdot}$ inside Q , $j_{n+1} = i+1$ so L_{i+1} enters $Z_i(\mathbf{w}_0)$ on the first (and projective) vertex $v_{\beta_{min}}$, directly from the border vertex l_{i+1} . In the other case, $\overset{i}{\cdot} \xrightarrow{\cdot} \overset{i+1}{\cdot}$ inside Q , L_{i+1} enters $Z_i(\mathbf{w}_0)$ on level n , and we can account for all vertices $L_k \cap L_{i+1}$. Either they lie on the right border of $Z_i(\mathbf{w}_0)$ if $k \leq i$, or are translates of vertices at levels $k = i+2, \dots, n$ of the left border of that set. \square

Proof of the Main Theorem 2.4

We shall proceed in two steps : first we establish a one-to-one correspondence $\pi \mapsto A(\pi)$ between GP-paths of type i and antichains of $P_i(Q)$, then fixing a GP-path π , we show that the vector \mathbf{k}_π has the same coefficients as the Lusztig move $\mathbf{l}_{A(\pi)}$.

A GP-path π of type i , consists by the oriented graph structure of $Y_i(\mathbf{w}_0)$ detailed in Figure 6.1, of three parts : A segment of $\delta_i^>$ from l_{i+1} up to entry into $Z_i(\mathbf{w}_0)$, a path π_{grid} with vertices belonging to $Z_i(\mathbf{w}_0)$ up to an exit vertex, and finally return to l_i on a segment of $\delta_i^<$. The first and last segments are uniquely defined by the first and last vertices of π_{grid} , thus the segment π_{grid} uniquely defines the whole path π .

Let us use the coordinate system given in Proposition 5.2 for $P_i(Q)$ which lies inside $H_i(Q)$. The order structure of $P_i(Q)$ is given by $(k_1, l_1) \leq (k_2, l_2)$ if and only if $k_1 \geq k_2$ and $l_1 \geq l_2$. All the vertices of $Z_i(\mathbf{w}_0)$ are by Propositions 5.2 and 5.4 of the same type : a crossing of a line L_{j_k} with $j_k \leq i$ which points backwards and upwards, with a line L_{j_l} with $j_l \geq i+1$ which points forwards and upwards. The path π_{grid} is therefore a staircase path. As such it is uniquely defined by the "extremities" of its steps, vertices which are a turning point from a forwards line into a backwards line. The inverse

image under Ψ of these vertices inside $P_i(Q)$ form a set $\{(k_1, l_1), (k_2, l_2), \dots, (k_r, l_r)\}$ whose elements have coordinates which are pairwise disjoint : for any $j \neq j'$, one has $k_j \neq k_{j'}$ and $l_j \neq l_{j'}$. By the nature of the order structure of $P_i(Q)$, this set is an antichain that we shall denote by $A(\pi)$.

Conversely, an antichain of $P_i(Q)$ is a set of vertices $\{(k_1, l_1), (k_2, l_2), \dots, (k_r, l_r)\}$ that have coordinates which are pairwise disjoint. The images of these vertices under Ψ form the extremities of a uniquely defined staircase path π_{grid} of $Z_i(\mathbf{w}_0)$. By adding the appropriate segments of the limiting path δ_i , we get a path $\pi(A)$ starting from l_{i+1} and returning to l_i . The nature of crossings inside $Z_i(\mathbf{w}_0)$ and Lemma 4.2 exclude the existence of forbidden crossings. The path $\pi(A)$ is therefore a GP-path of type i .

The mapping $A \mapsto \pi(A)$ is the inverse of $\pi \mapsto A(\pi)$, and vice-versa, so we have a one-to-one correspondence ψ between GP-paths of type i and antichains of $P_i(Q)$.

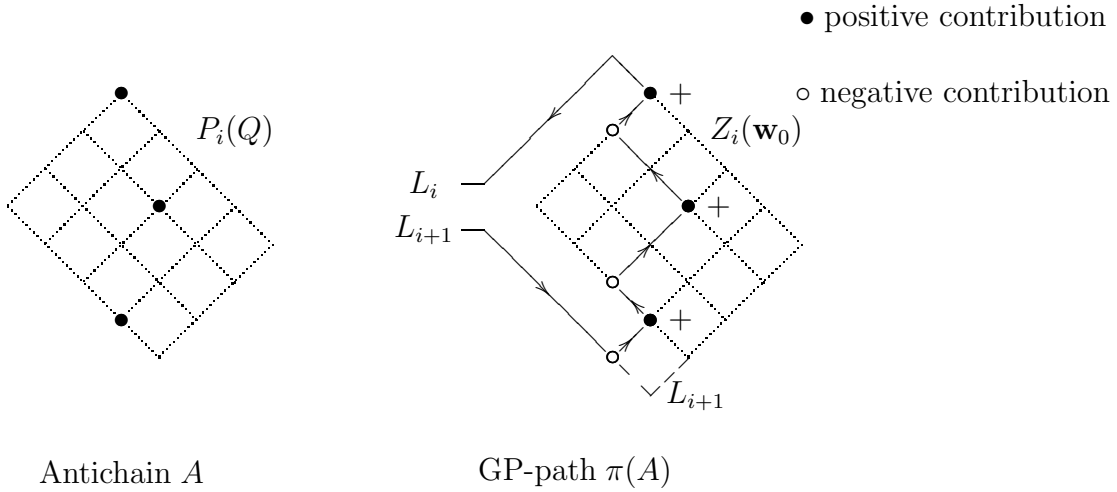


Figure 6.2 : Antichain of $P_i(Q)$ and corresponding GP-path of type i

Let us fix now a GP-path, and study its contributing vertices. Recall a vertex v_{β_h} contributes a term $+t_h$ in $\mathbf{k}_\pi \cdot \mathbf{t}$ if π changes strands $L_j \rightarrow L_k$ at v_{β_h} with a decrease from j to k , and a term $-t_h$ if a change of strands $L_j \rightarrow L_k$ occurs at v_{β_h} with an increase from j to k . If π stays on the same strand while passing through v_{β_h} , no contribution occurs.

In view of Figure 6.2, a change of strands occurs at an extremal vertex v_{β_h} of π_{grid} . At this vertex, the incoming strand L_{j_k} verifies $j_k \geq i + 1$, while the outgoing strand L_{j_l} verifies $j_l \leq i$. We see a decrease of indices occurs in the passage $L_{j_k} \rightarrow L_{j_l}$, so v_{β_h} is a positively contributing vertex.

Let us denote by J_π the order ideal defined by the extremal vertices of π_{grid} , inside $Z_i(\mathbf{w}_0)$ endowed with the poset structure induced by that $P_i(Q)$. Consider a minimal element $v_{\beta_{h'}}$ of $Z_i(\mathbf{w}_0) \setminus J_\pi$, and apply the translation τ_{wd} . We get a vertex $\tau_{wd}(v_{\beta_{h'}})$ of π with the following possible cases :

- i) $\tau_{wd}(v_{\beta_{h'}})$ is in $Z_i(\mathbf{w}_0)$, in which case it is adjacent to three chambers of $\mathcal{Z}_i(\mathbf{w}_0)$. A passage from a backward oriented line L_{j_k} to a forward oriented line L_{j_l} occurs, with an increase of indices.

- ii) $\tau_{wd}(v_{\beta_{h'}})$ is a vertex of $\delta_i^>$, outside $Z_i(\mathbf{w}_0)$. The strand L_{i+1} is of minimal index among forwardly oriented strands, so an increase of indices occurs by Proposition 6.1.
- iii) $\tau_{wd}(v_{\beta_{h'}})$ is a vertex of $\delta_i^<$ outside $Z_i(\mathbf{w}_0)$. The strand L_i is of maximal index among backwards oriented strands, so again, by Proposition 6.1 an increase of indices occurs.

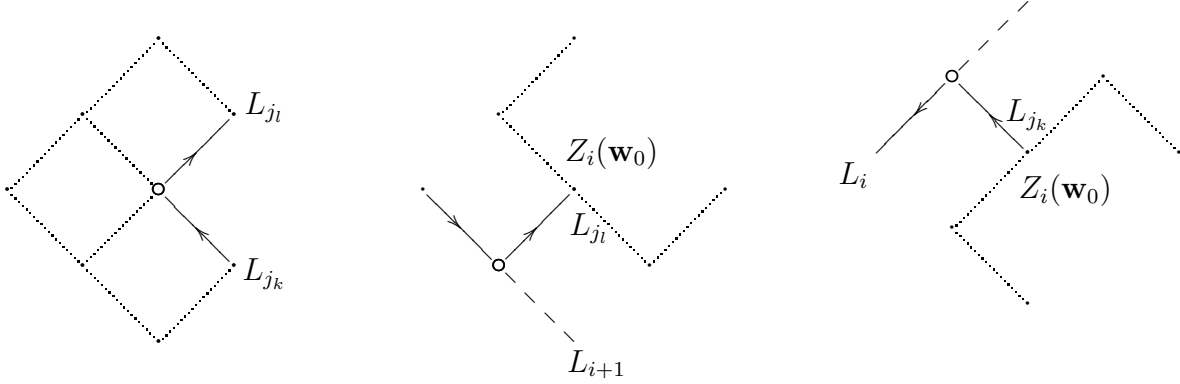


Figure 6.3 : Negative contributing vertices

We see that the translation $\tau_{wd}(v_{\beta_{h'}})$ results in all cases with a negative contribution -1 to \mathbf{k}_π .

Finally, if v_{β_h} is neither maximal inside J_π , nor a translate of a minimal element of $Z_i(\mathbf{w}_0) \setminus J_\pi$, π stays on the same strand while passing through it, so that no contribution occurs.

Comparing this discussion with the definition of the Lusztig move out of the antichain $A(\pi)$ we see we get exactly the same positive and negative contributions $\pm t_h$ in both cases. \square

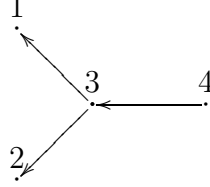
Remarks

i) The staircase paths used in the proof seem closely related to Le-diagrams defined by A. Postnikov in his study of positroids (compare Figure 6.2 with [17] Figure 17.3).

ii) The neat structure of $Y_i(\mathbf{w}_0)$ described by Figure 6.1 breaks down for reduced expressions \mathbf{w}_0 of type A_n that are not adapted to a quiver. Its subset $Z_i(\mathbf{w}_0)$ may include vertices which lie outside of $H_i(\mathbf{w}_0)$. Furthermore, the regular grid structure of $Z_i(\mathbf{w}_0)$ is destroyed by the appearance of horizontal adjacencies between vertices of $Z_i(\mathbf{w}_0)$.

7. A D_n example

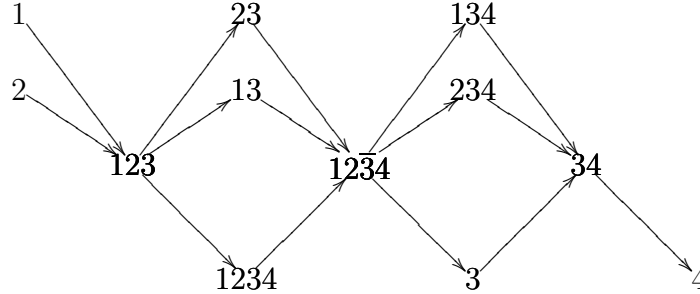
Let us consider the D_4 type quiver Q , given in the appendix of [18] as satisfying condition (L) :



Let us verify the conjecture stated in section 2. $\mathbf{w}_0 = s_1 s_2 s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4 s_3 s_4$ is adapted to Q , with reflection ordering

$$\begin{array}{lll} \beta_1 = \alpha_1 & \beta_5 = \alpha_1 + \alpha_3 & \beta_9 = \alpha_2 + \alpha_3 + \alpha_4 \\ \beta_2 = \alpha_2 & \beta_6 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \beta_{10} = \alpha_3 \\ \beta_3 = \alpha_1 + \alpha_2 + \alpha_3 & \beta_7 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 & \beta_{11} = \alpha_3 + \alpha_4 \\ \beta_4 = \alpha_2 + \alpha_3 & \beta_8 = \alpha_1 + \alpha_3 + \alpha_4 & \beta_{12} = \alpha_4 \end{array}$$

Lusztig parametrization is given by elements $\mathbf{t} \in \mathbb{N}^{12}$ with t_i corresponding to $[\beta_i]$. We shall concisely denote a vertex $[\beta]$ of Γ_Q by the indexes of the simple roots appearing. For instance 134 stands for $[\alpha_1 + \alpha_3 + \alpha_4]$. We shall denote the vertex $[\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4]$ by $12\bar{3}4$. The Auslander-Reiten quiver is given by



The $P_i(Q)$ sets are :

$$\begin{aligned} P_1(Q) &= \{1\} \\ P_2(Q) &= \{2\} \\ P_3(Q) &= \{123, 23, 13, 12\bar{3}4, 3\} \\ P_4(Q) &= \{1234, 12\bar{3}4, 134, 234, 34, 4\}. \end{aligned}$$

There are respectively 1, 1, 6, 7 antichains of types 1, 2, 3, 4.

Kashiwara's embedding corresponds to [4] Theorem 5.10. The computation of the set $K_{\mathbf{w}_0}^{BZ}$ requires therefore, by [4] Proposition 3.3 (iii), the use of the reduced expression \mathbf{w}_0^{op} , whose reflection ordering is reversed as compared with that of \mathbf{w}_0 (α_4 occurs first, α_1 occurs last), as well as a reversal in the numbering of coordinates. In our case, the set of indices of \mathbf{w}_0^{op} is $\mathbf{i}^{op} = (4, 3, 4, 2, 1, 3, 4, 2, 1, 3, 2, 1)$.

Fix $i \in I$, and let $E(\omega_i)$ the corresponding fundamental representation of $U_q(\mathfrak{g})$ of type D_4 . An \mathbf{i}^{op} -trail π of type i goes from ω_i to $w_0 s_i \omega_i$. It is given by a set

of coefficients $\mathbf{m} = (m_1, m_2, \dots, m_{12})$ such that the monomial $e_1^{m_1} e_2^{m_2} \dots e_{12}^{m_{12}}$ induces a non-zero mapping from the weight space $E(\omega_i)_{w_0 s_i \omega_i}$ to the highest weight space $E(\omega_i)_{\omega_i}$. π defines a sequence of weights $\omega_i = \gamma_0, \gamma_1, \dots, \gamma_{12} = w_0 s_i \omega_i$ with

$$\gamma_k := \gamma_0 - \sum_{l=1}^k m_l \alpha_{i_l}.$$

The trail π defines a vector $\mathbf{h}_\pi \in \mathbb{Z}^{12}$ whose k -th coordinate is $h_k := (\frac{\gamma_{k-1} + \gamma_k}{2}, \alpha_{i_k})$, ($k = 1, \dots, 12$) ([4] page 5, (2.2)). Let us denote $\mathbf{k}_\pi := (h_{12}, h_{11}, \dots, h_1)$. Then, by [4] Theorem 3.10, the set $K_{\mathbf{w}_0}^{BZ}$ of all vectors \mathbf{k}_π , where π is any \mathbf{i}^{op} -trail, of any type i defines $\mathcal{C}_{\mathbf{w}_0}$.

Let us consider the \mathbf{i}^{op} -trail π of type 3 with coefficients $(0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1)$. The list of weights γ_k through which π passes, as well as the coordinates h_k , are given by the following table (weights are expressed by their coordinates with respect to the basis of fundamental weights) :

k	γ_k	$(\gamma_{k-1} + \gamma_k)/2$	α_{i_k}	h_k
0	$[0, 0, 1, 0]$			
1	$[0, 0, 1, 0]$	$[0, 0, 1, 0]$	α_4	0
2	$[1, 1, -1, 1]$	$[\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}]$	α_3	0
3	$[1, 1, 0, -1]$	$[1, 1, -\frac{1}{2}, 0]$	α_4	0
4	$[1, -1, 1, -1]$	$[1, 0, \frac{1}{2}, -1]$	α_2	0
5	$[-1, -1, 2, -1]$	$[0, -1, \frac{3}{2}, -1]$	α_1	0
6	$[0, 0, 0, 0]$	$[-\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}]$	α_3	1
7	$[0, 0, 1, -2]$	$[0, 0, \frac{1}{2}, -1]$	α_4	-1
8	$[0, 0, 1, -2]$	$[0, 0, 1, -2]$	α_2	0
9	$[0, 0, 1, -2]$	$[0, 0, 1, -2]$	α_1	0
10	$[1, 1, -1, -1]$	$[\frac{1}{2}, \frac{1}{2}, 0, -\frac{3}{2}]$	α_3	0
11	$[1, -1, 0, -1]$	$[1, 0, -\frac{1}{2}, -1]$	α_2	0
12	$[-1, -1, 1, -1]$	$[0, -1, \frac{1}{2}, -1]$	α_1	0

We get $\mathbf{h}_\pi = (0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0)$, hence $\mathbf{k}_\pi = (0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0)$, which defines the string cone inequality $t_7 - t_6 \geq 0$. If we consider the antichain $A = \{12\bar{3}4\}$ of $P_3(Q)$, then $V_A = [\beta_7]$, $U_A = [\beta_6]$, so we get $\mathbf{l}_A = \mathbf{k}_\pi$.

The enumeration of all possible \mathbf{i}^{op} -trails in our case (with the help of quagroup package [7]) shows there is a one-to-one correspondence between antichains A of $P_i(Q)$ and \mathbf{i}^{op} -trails π of type i , for each of the types $i = 1, \dots, 4$. For every antichain A there is a unique trail π with $\mathbf{l}_A = \mathbf{k}_\pi$. We give these correspondences in the table below. The first column is the set of defining inequalities of $\mathcal{C}_{\mathbf{w}_0}$. Next to each inequality is the antichain A defining the corresponding Lusztig move \mathbf{l}_A , and the \mathbf{i}^{op} -trail leading to the corresponding vector $\mathbf{k}_\pi \in K_{\mathbf{w}_0}^{BZ}$.

Type	Inequality	Antichain	\mathbf{i}^{op} -trail position	\mathbf{i}^{op} -trail coefficients \mathbf{m}
1	$t_1 \geq 0$	1	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,0,0,0,1,1,1,1,0,1,0,0)$
2	$t_2 \geq 0$	2	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,0,0,1,0,1,1,0,1,1,0,0)$
3	$t_3 - t_1 - t_2 \geq 0$	123	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,1,1,1,1,2,1,1,1,0,0,0)$
	$t_4 - t_2 \geq 0$	23	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,1,1,1,1,2,1,1,0,0,0,1)$
	$t_5 - t_1 \geq 0$	13	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,1,1,1,1,2,1,0,1,0,1,0)$
	$t_4 + t_5 - t_3 \geq 0$	13, 23	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,1,1,1,1,2,1,0,0,0,1,1)$
	$t_7 - t_6 \geq 0$	$12\bar{3}4$	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,1,1,1,1,1,1,0,0,1,1,1)$
	$t_{10} \geq 0$	3	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,1,0,1,1,1,2,0,0,1,1,1)$
4	$t_6 - t_3 \geq 0$	1234	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(1,1,0,1,1,1,0,0,0,0,0,0)$
	$t_7 - t_4 - t_5 \geq 0$	$12\bar{3}4$	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(1,1,0,1,1,0,0,0,0,1,0,0)$
	$t_8 - t_5 \geq 0$	134	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(1,1,0,1,0,0,0,0,1,1,0,0)$
	$t_9 - t_4 \geq 0$	234	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(1,1,0,0,1,0,0,1,0,1,0,0)$
	$t_8 + t_9 - t_7 \geq 0$	134, 234	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(1,1,0,0,0,0,0,1,1,1,0,0)$
	$t_{11} - t_{10} \geq 0$	34	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(1,0,0,0,0,1,0,1,1,1,0,0)$
	$t_{12} \geq 0$	4	$4\ 3\ 4\ 2\ 1\ 3\ 4\ 2\ 1\ 3\ 2\ 1$	$(0,0,1,0,0,1,0,1,1,1,0,0)$

Remark : The trail π of type 3 considered above in detail, passes through the weight $[0, 0, 0, 0]$. All other trails of the table pass only through extremal weights. The existence of π shows that condition (L) is not strong enough to allow an analogue of [4] Theorem 3.14. The subexpression corresponding to π is not a reduced word.

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